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ON THE AVERAGE TRANSFER COEFFICIENT IN PERIODIC HEAT EXCHANGE—II

SEMI-INFINITE SOLID BODY

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Abstract—In extension of a previous study this work describes periodic heat exchange between a gas and a semi-infinite solid body. The gas phase undergoes a stepwise change in temperature with the step length as a free variable. A boundary condition of the third kind is considered and for a time-independent heat-transfer coefficient an analytic solution to the problem is developed.

A more realistic situation, where the transfer coefficient changes with gas temperature, is solved by a forward-marching process. These numerical results are correlated on the basis of a suitable average heat-transfer coefficient to be employed in the analytic solution. Thereby the results become easily adaptable to further techniques aimed at optimizing periodic exchange processes.

To H. Hausen in recognition of his pioneering work

NOMENCLATURE

- h, fluid–solid heat-transfer coefficient [$Wm^{-2}K^{-1}$];
- k, thermal conductivity of solid [$Wm^{-1}K^{-1}$];
- q, heat-flux density [Wm^{-2}];
- t, time [s];
- T, temperature [K], referring to solid if without subscript;
- U, overall heat-transfer coefficient [$Wm^{-2}K^{-1}$];
- y, coordinate from solid surface to interior [m];
- α, thermal diffusivity of solid [$m^2 s^{-1}$];
- ϕ, phase angle of heating [rad];
- ω, oscillation frequency [s^{-1}].

Dimensionless quantities

- Bi, $= h\sqrt{(\alpha/\omega)}/k$ Biot number;
- Bi^*, $= Bi\sqrt{\left(\dfrac{\pi^2}{\phi(2\pi-\phi)}\right)}$ corrected Biot number;
- H, $= \dfrac{q}{h(T_{g2}-T_{g1})}$ heat-flux parameter;
- j, n, integer;
- Y, $= y/\sqrt{(\alpha/\omega)}$ depth coordinate;
- Δ_s, difference between extreme solid temperatures according to equation (7);
- θ, solid temperature;
- $\bar{\theta}_s$, arithmetic average of extreme solid temperatures according to equation (8);
- τ, time coordinate.

Subscripts

- e, exponential;
- g, fluid;
- h, harmonic;
- 1, during cooling;
- 2, during heating.

1. INTRODUCTION

PERIODIC heat exchange between a fluid and a solid phase has been studied extensively in connection with the design and performance of thermal regenerators. However, the analytic treatment still suffers from the fact that the general problem is three-dimensional, i.e. the solid temperature varies with time, depth from the solid surface and in the direction of gas flow. Usually, either the second [1] or both second and third variable [2] are subjected to some approximation procedure before the final solution is obtained. A brief summary of the state-of-the-art has been given recently [3]. In view of this background the present study neglects the third variable completely which would apply to an infinitely short regenerator or to a gas flow of infinite thermal capacity.

On the other hand there are many processes where a solid body is merely heated and cooled periodically over its entire exchange surface. The outside wall of a building, the brick lining of a rotary kiln or the work roll of a hot strip mill are just a few examples where one is interested in either the storage capacity or the extreme solid temperatures or both. On the basis of a thermal analysis the important parameters can be varied towards an optimum design of the element or apparatus. The inside wall temperature and heat-flux distribution determines the heating policy in a building [4], the periodic heat flux from the charge to the lining and from the lining to the gas affects the length of the kiln required for a certain duty [5, 6] and the maximum surface temperature of the work roll decides on the amount of additional cooling [7].

The theoretical study of such problems can be done under various assumptions about the nature of the boundary condition or the solid body itself. With a boundary condition of the first kind, i.e. known periodic solid–surface temperature and for a semi-infinite solid body the solution is well-established [8];

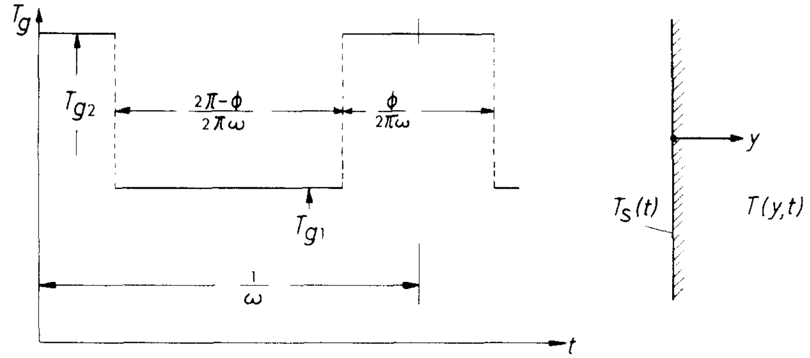

FIG. 1. Illustration of the general problem.

the same applies to a boundary condition of the second kind or known periodic flux at the surface [9, 10]. However, often it is certainly more realistic to consider a boundary condition of the third kind which means specifying the periodic fluid temperature together with the heat-transfer coefficient and solving for the solid temperature. This problem was analyzed for a sinusoidal fluid temperature distribution, constant heat-transfer coefficient and solid body of any thermal thickness [11]. Unfortunately, in many technical situations the fluid temperature is not sinusoidal and, to complicate things even more the heat-transfer coefficients during heating and cooling differ markedly from each other. The latter may be due to either different temperature levels only or to the presence of two different fluid phases as is the case for the rotary kiln. A straightforward numerical study of each specific problem could in principle yield the required results, but we believe that a stepwise change in fluid temperature and heat-transfer coefficient together with a variable step length of heating is sufficiently realistic to justify a general treatment similar to that of Groeber [11].

In a previous article [3] we have considered the solid body without thermal resistance and subjected it to the above boundary condition. Following a similar approach we now solve the problem for the semi-infinite solid: a general analytic solution for a constant heat-transfer coefficient and a variable phase angle of heating is developed first; then follows a second, in this case numerical approach in which the heat-transfer coefficient varies simultaneously with the fluid temperature. The comparison of both results leads to an average heat-transfer coefficient for use in the general analysis. The advantage of this aforegoing over other possible ones is that we eventually obtain a closed-form solution of physically correct structure although the problem, to our knowledge, cannot yet be solved analytically.

It is noted that we prefer not to present the general problem of the finite solid body of finite thermal conductivity; the mathematical approach is much the same but the equations become prohibitively voluminous without providing significantly more physical insight. The really valuable information in that case is the temperature oscillation in the symmetry axis of the body; its relative magnitude decides on the applicability of this or the previous [3] analysis. However, once the general approach is established the main results of the general problem will be presented in a brief follow up.

2. FORMULATION OF THE PROBLEM

It is assumed that the semi-infinite wall is homogeneous and that its physical properties are independent of temperature. Then the extreme temperatures and storage capacity are evaluated from the solution to the following Fourier equation (see Fig. 1 and the Nomenclature):

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial y^2}. \tag{1}$$

The boundary condition on the exchange surface $y = 0$ reads

$$k\frac{\partial T_s}{\partial y} = h(T_s - T_g), \tag{2}$$

where both the gradient and the value of T_s are unknown. The second boundary condition simply states that at $y = \infty$ the temperature oscillation has disappeared, i.e.

$$T|_{y=\infty} = \text{constant}. \tag{3}$$

As in any periodic problem, where one is not interested in the transient behaviour, the initial condition is expressed in the time-dependent boundary condition (2). It is pointed out that this equation becomes nonlinear when the heat-transfer coefficient varies with time. For this reason the analytic treatment is limited to a constant transfer coefficient.

The fluid temperature T_g is expressed in terms of a Fourier series, hence

$$T_g(t) = T_{g1} + (T_{g2} - T_{g1}) \times \left[\frac{\phi}{2\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{\sin(n\phi/2)}{n}\cos(2\pi n\omega t)\right\}\right]. \tag{4}$$

This is inserted in equation (2) whereby

$$\theta = \frac{T - T_{g1}}{T_{g2} - T_{g1}} - \frac{\phi}{2\pi}$$

emerges as an obvious dimensionless temperature. The inclusion of the integrated average of the fluid temperature is feasible as long as the heat-transfer coefficient is the same during heating and cooling; then the integrated average of the solid temperature is the same.

The reference length is chosen in analogy to the "penetration thickness" commonly employed in transient problems; it is in fact an inverse Fourier number modified for the periodic case. With

$$Y = y/\sqrt{(\alpha/\omega)} \quad \text{and} \quad \tau = t\omega$$

the transformed problem becomes

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial Y^2}, \tag{1a}$$

$$\frac{\partial \theta_s}{\partial Y} = Bi\left[\theta_s - \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{\sin(n\phi/2)}{n}\cos(2\pi n\tau)\right\}\right] \tag{2a}$$

and

$$\theta|_{Y=\infty} = 0. \tag{3a}$$

Here we have introduced the Biot number

$$Bi = \frac{h}{k}\sqrt{\left(\frac{\alpha}{\omega}\right)}$$

to account for varying physical properties of the solid as well as frequency of the temperature oscillation. Again, it is seen that this is a true parameter only if the heat-transfer coefficient stays constant or if at least a time-independent average can be used. Equation (3a) demonstrates the importance of Y rather than y for the analysis to be applicable; the solid does not have to be infinitely thick in a geometric sense.

3. ANALYTIC SOLUTION FOR CONSTANT HEAT-TRANSFER COEFFICIENT

The solution of equation (1a) subject to the boundary conditions (2a) and (3a) can be obtained in different ways. Being interested only in the periodic steady state we prefer to employ the straightforward method of complex temperature as outlined elsewhere [12]. The development is given in Appendix A and as a final result the solid temperature becomes

$$\theta(\tau, Y) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{\sin(n\phi/2)}{n} \times \exp[-Y\sqrt{(\pi n)}](C_1\cos[2\pi n\tau - Y\sqrt{(\pi n)}] + C_2\sin[2\pi n\tau - Y\sqrt{(\pi n)}])\right\} \tag{5}$$

where

$$C_1 = \frac{1+\sqrt{(\pi n)}/Bi}{1+2\sqrt{(\pi n)}/Bi + 2\pi n/Bi^2}$$

and

$$C_2 = \frac{\sqrt{(\pi n)}/Bi}{1+2\sqrt{(\pi n)}/Bi + 2\pi n/Bi^2}.$$

The combination of the two trigonometric functions to one with argument

$$\left\{2\pi n\tau - Y\sqrt{(\pi n)} - \tan^{-1}\left[\frac{1}{1+Bi/\sqrt{(\pi n)}}\right]\right\}$$

would be misleading because it was shown previously [3] that with a stepwise change in fluid temperature there is no time lag between fluid and solid–surface temperature.

The further processing of equation (5) depends on the considered applications. In general, one would be interested in the extreme solid temperatures, to be found on the surface, as well as in the amount of energy stored and released per cycle or per unit time.

Extreme temperatures

In the symmetric problem [11] the difference between maximum and minimum surface temperature is sufficient to find the extreme values whereas here the arithmetic mean or some other reference value is required in addition. For two combinations of phase angle of heating and Biot number the same difference between maximum and minimum surface temperature may result but the absolute values could be quite different.

The surface temperature is given by

$$\theta_s(\tau) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{\sin(n\phi/2)}{n} \times [C_1\cos(2\pi n\tau) + C_2\sin(2\pi n\tau)]\right\}. \tag{6}$$

From the above remarks and Fig. 1 we conclude further that

$$\theta_{s,\max} - \theta_{s,\min} = \theta_s|_{(\phi/4\pi\omega)} - \theta_s|_{(4\pi-\phi/4\pi\omega)} \equiv \Delta_s$$

and

$$\Delta_s \equiv \frac{T_{s,\max} - T_{s,\min}}{T_{g2} - T_{g1}} = \frac{4}{(\sqrt{\pi})Bi}\sum_{n=1}^{\infty}\left\{\frac{\sin^2(n\phi/2)}{(\sqrt{n})[1+2\sqrt{(\pi n)}/Bi + 2\pi n/Bi^2]}\right\}. \tag{7}$$

It is seen that, although equation (7) is of a particularly simple structure, the convergence of the series is poor especially when the Biot number is large. The reason for this becomes obvious when one considers the surface-temperature profile according to equation (6); for illustration two different cases are plotted in Fig. 2. At the switching point the heat flux must be discontinuous because it changes sign. There results a kink in the surface temperature which is difficult—though not impossible—to represent by a Fourier series. The latter tends to round off the corner and with larger Biot number an increasing number of terms has to be considered in the series. Equation (7) is represented in Fig. 3 showing the effect of variable phase angle of heating on the surface-temperature oscillation. The series was taken up to 3000 terms for $Bi \leqslant 2$ and up to 8000 terms for larger Biot numbers. It may be pointed out that a straightforward evaluation of the Fourier series breaks down at such large arguments of the trigonometric series; instead, some recurrence relations [13] had to be used so that an efficient solution could be generated without any loss in computation accuracy. Still, the comparison with numerical results, to be discussed later, shows a discrepancy of up to 4% which is equivalent to a −2% error in the maximum temperature. On the other hand, such an error only occurs at large Biot numbers where

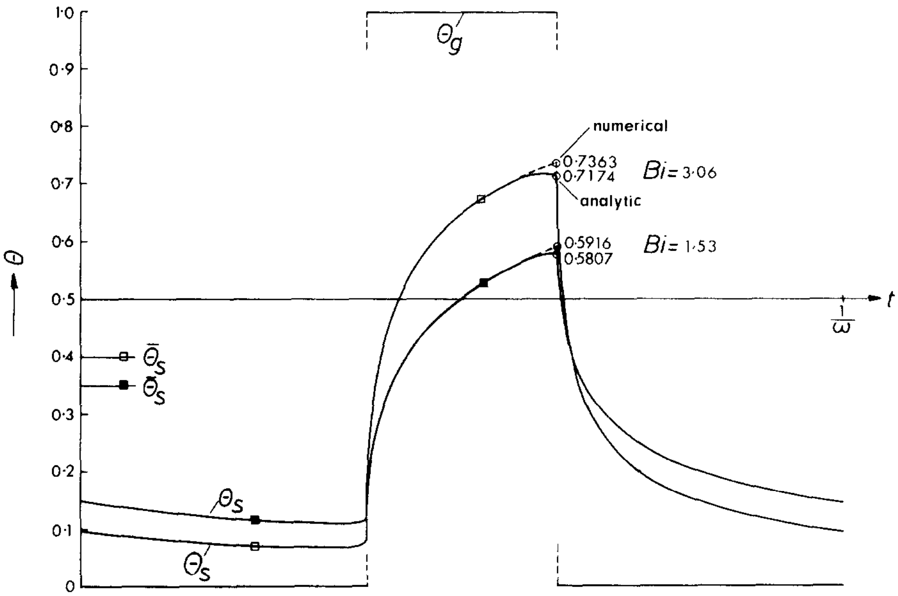

FIG. 2. Solid–surface temperature in the periodic steady-state ($\phi = \pi/2$, $h_1 = h_2$); comparison between analytic and numerical solution.

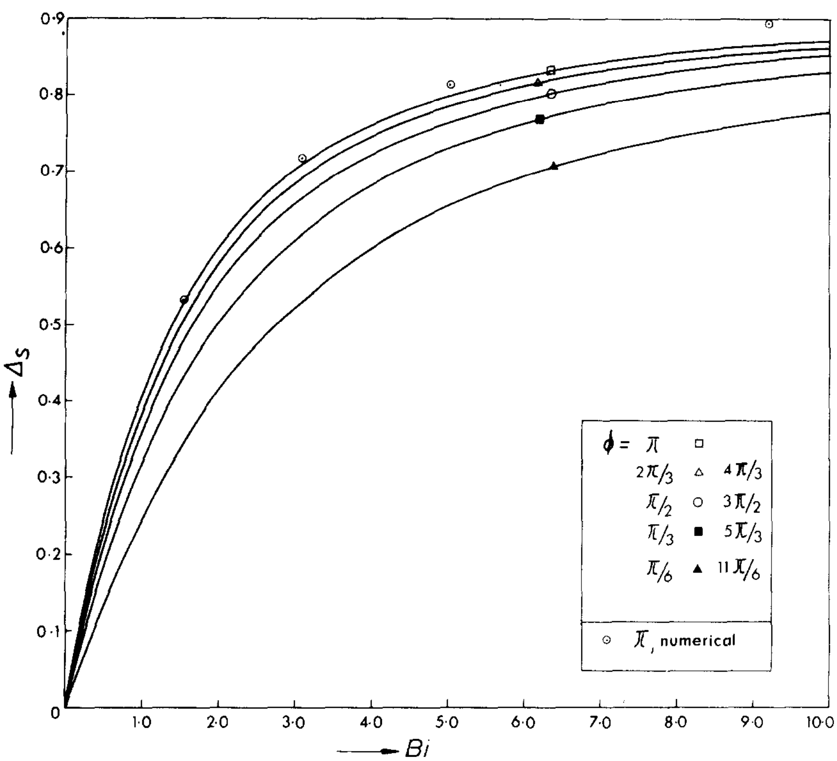

FIG. 3. Difference between maximum and minimum solid temperature at uniform heat-transfer coefficient and with relative heating time as parameter.

the maximum solid temperature is close to the fluid temperature and a slight under-estimation is seldom critical.

When comparing Fig. 3 with the corresponding results for the thermally thin solid [3] one has to bear in mind the different definitions of the Biot number. In spite of the same physical meaning of both parameters a numerical comparison is inadmissible because they do not employ the same variables. However, a qualitative comparison as to the influence of the phase angle of heating demonstrates a major difference between the two systems. Here, the change in amplitude with decreasing ϕ is much smaller than in [3] which is explained as follows: having a certain resistance to absorb energy the solid needs some time to do so; during this time a solid layer next to the surface will settle very quickly at a temperature such that the flux of energy arriving from the fluid can be accommodated in the interior. Hence a decrease in ϕ would only cut off the tail of the temperature–time curve where changes are small. This is not so with a slab of infinite thermal conductivity. There the temperature changes exponentially which means that differential changes are significant over the entire heating period as long as Δ_s is not close to one. Consequently, a decrease in ϕ has a more direct effect on the final temperature.

On the other hand, we conclude from this difference in behaviour that neither an exponential nor a harmonic average of the heat-transfer coefficient is suitable to simplify the analytic result. We recall from [3] that with the exponential average of h, inserted in the Biot number, all the curves for $\phi < \pi$ were projected onto the curve for $\phi = \pi$. It is verified easily that, in the present case, errors of 40% or more could result from using either of the two averages. This may demonstrate that the commonly considered concept of a harmonic average is not feasible.

In order to evaluate extreme temperatures it is convenient to use the arithmetic mean of maximum and minimum temperature together with equation (7). From equation (6) we find that

$$\bar{\theta}_s \equiv \frac{\dfrac{T_{s,\max}+T_{s,\min}}{2} - T_{g1}}{T_{g2}-T_{g1}} = \frac{\phi}{2\pi} + \sum_{n=1}^{\infty} C_1 \frac{\sin n\phi}{n\pi}. \quad (8)$$

Here the errors at the times of switch-over are subtracted which is the reason for the improved convergence of the series. Equation (8) is easily evaluated and as the results are symmetric to $\bar{\theta}_s = 0.5$ only the lower half is shown in Fig. 4. It is obvious that e.g.

$$\bar{\theta}_s(\phi = \tfrac{3}{2}\pi) = 1 - \bar{\theta}_s(\phi = \pi/2).$$

For large Biot numbers

$$\bar{\theta}_s \to 0.5 \quad \text{since} \quad \Delta_s \to 1,$$

whereas in the limit of

$$Bi \to 0 \quad \text{then} \quad \bar{\theta}_s \to \frac{\phi}{2\pi},$$

the integrated average of the fluid temperature. The latter result is evident because the integrated averages of fluid and solid temperature must be the same as long as h is constant. With decreasing fluctuation, on the other hand, arithmetic and integrated average approach each other.

Storage capacity

The energy released per unit time is given by

$$q = -k\omega \int_{(\phi/4\pi\omega)}^{(4\pi-\phi/4\pi\omega)} \left.\frac{\partial T}{\partial y}\right|_{y=0} \mathrm{d}t. \quad (9)$$

In terms of the dimensionless quantities introduced earlier this becomes

$$H \equiv \frac{q}{h(T_{g2}-T_{g1})} = -\frac{1}{Bi}\int_{(\phi/4\pi)}^{(4\pi-\phi/4\pi)} \left.\frac{\partial\theta}{\partial Y}\right|_{Y=0} \mathrm{d}\tau. \quad (9a)$$

The heat-flux parameter H, which was already introduced previously [3], can be interpreted as a correction factor to the heat-transfer coefficient. H reaches a maximum value at infinite rotational frequency ω because then the process becomes equivalent to the indirect heat-exchange process with fluids at T_{g2} and T_{g1} and heat-transfer coefficients h (see [14]). In that case

$$q = h(T_{g2} - T_s) = h(T_s - T_{g1}) \quad (10)$$

or

$$q = U(T_{g2} - T_{g1})$$

with

$$U = \frac{1}{\dfrac{2\pi}{h\phi} + \dfrac{2\pi}{h(2\pi-\phi)}} = h\frac{\phi(2\pi-\phi)}{4\pi^2}.$$

Therefore

$$H_{Bi\to 0} = \frac{\phi(2\pi-\phi)}{4\pi^2} \quad (11)$$

which is largest in the symmetric case, i.e.

$$H_{Bi\to 0}(\phi = \pi) \equiv H_{\max} = 0.25.$$

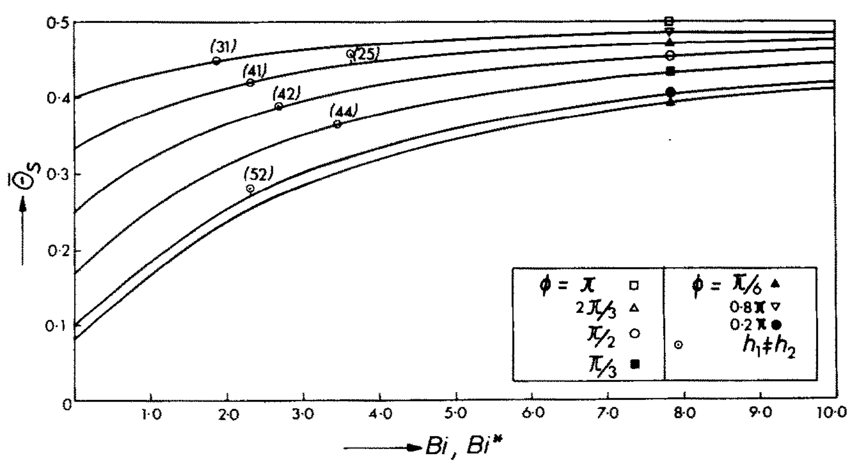

FIG. 4. Arithmetic average of maximum and minimum solid temperature; the curves are symmetric to $\theta_s = 0.5$.

Smaller or larger values of ϕ result in a reduced value of H which agrees with the concept of the harmonic average. When $Bi > 0$ then H must decrease further because the two processes given by equation (10) do not occur simultaneously any more.

From the definition of the Biot number we also find that

$$Bi \to 0 \quad \text{for} \quad k \to \infty;$$

hence in the limit this analysis should yield the same result as the previous one [3]. In fact, it was shown there that the same equation (11) holds for the solid of negligible thermal resistance.

based on a modified heat-transfer coefficient (see [15]) is preferential in the symmetric problem but turns out to be less convenient in the more general problem treated here.

4. NUMERICAL SOLUTION OF THE GENERAL PROBLEM

As pointed out before the strict analytic treatment of the problem is limited to a linear boundary condition (2a). It seems that with a time-dependent heat-transfer coefficient either an approximate analytic or a numerical method has to be employed. The majority of the former have, however, only been developed for transient problems with a well-defined initial condition

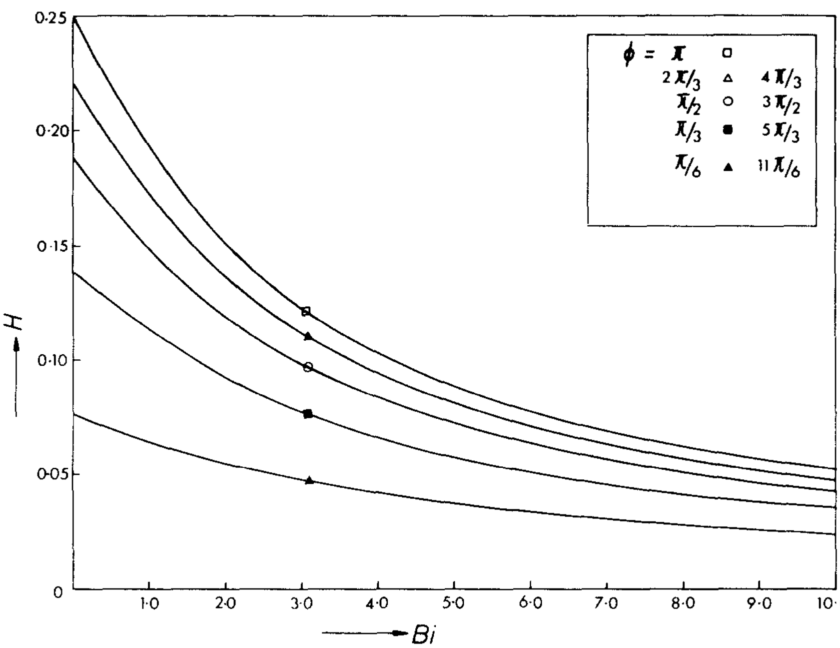

FIG. 5. Dimensionless storage capacity of solid body at uniform heat-transfer coefficient.

The evaluation of the integral in equation (9a) is lengthy but elementary and may be omitted. The final result is

$$H = \frac{2}{\pi(\sqrt{\pi})Bi}\sum_{n=1}^{\infty}\left\{\frac{(\sqrt{n})[1+2\sqrt{(\pi n)}/Bi]\sin^2(n\phi/2)}{n^2[1+2\sqrt{(\pi n)}/Bi+2\pi n/Bi^2]}\right\} \quad (12)$$

and is illustrated in Fig. 5. It is seen that with all other variables staying constant an infinite rotational frequency yields a maximum storage capacity. This result has some impact on the design of rotary heat-exchange equipment because it will lead the way towards establishing optimum operating conditions for such processes.

An unfortunate feature of the chosen set of parameters is that h appears in both the Biot number and the heat-flux parameter. From Fig. 5 one may conclude that with increasing h and hence Bi the storage capacity would drop whereas, in fact, the opposite is found by evaluating q at two different Biot numbers. However, it was preferred to operate with established and directly applicable parameters rather than introduce new ones. The alternative concept of defining a Nusselt number

[16, 17]. This is also the case with the heat-balance-integral method [18] which is essentially a sophistication of the Ritz–Galerkin scheme [19]; although it should be possible to modify this method for use in the periodic problem a reasonable accuracy would lead to rather complicated approximation functions. Another approach discussed recently [20] and directly applicable to periodic problems again requires that the boundary condition be linear in time.

A disadvantage of the straightforward numerical methods is that the entire transient behaviour has to be evaluated in time and space; the periodic steady state is reached when no significant changes in solid temperature are experienced from one cycle to the next. Although the initial condition, i.e. the temperature level at zero time can in principle be arbitrary it still has a major effect on the required computation time. If this level is unknown then both the explicit (finite-difference) and the implicit (Crank–Nicolson) methods may become prohibitively time-consuming. Even with symmetric problems such a direct approach is often extremely expensive [21]. Jaeger [22] suggested a method based on the superposition of incremental analytic solutions. For a linear boundary condition only a set of algebraic equations needs to be solved in order to yield an overall analytic solution, but for the non-linear boundary condition an iterative numerical technique has to be applied which, adapted to our problem, becomes rather involved.

Finally, a procedure suggested by Bentwich [23] may be discussed briefly; it is essentially a Fourier-series approach to the complete problem. By developing the heat-transfer coefficient h as well as the fluid and solid-surface temperatures into Fourier series and breaking off the series for h after n terms one obtains from boundary condition (2a) a set of n algebraic equations. From these the coefficients of the surface-temperature series are determined. By this aforegoing the transient stage is eliminated; in addition, the truncated series for (hT_g) may, for many applications, be more realistic because a mathematic step is unlikely to occur in a real process. The only restriction is that n would be limited to values $\leqslant 10$ in order to keep the calculation effort within reasonable limits. However, a direct comparison with the analytic results of Section 3 then becomes difficult because this approach may lead to substantially reduced values of Δ_s. Even with a constant value of n the relative accuracy will not be the same for all values of ϕ and Bi so that the specification of a suitable average of h to be used in equation (7) becomes unreliable. Therefore, in spite of its merits, this procedure was not adopted for the present study.

Instead, a modified numerical technique, outlined in Appendix B, is used. Having started with an elaborate finite-difference scheme we realized that the correct initial temperature level in the slab is absolutely essential for convergence to periodic steady state within a reasonable time. The reason for this is quite simple: once the wrong level has been chosen only the difference between absorbed and released energy is available for adjustment within one cycle. However, the solid is per assumption of infinite "thermal" thickness so that a large number of cycles is needed to achieve a significant rise or drop in the temperature level (theoretically an infinite number of cycles is necessary). As small time and space intervals have to be chosen for reasons of accuracy and stability also an iterative procedure on the basis of, e.g. ten cycles had to be abandoned, particularly in view of the required amount of results (see Fig. 7). In conclusion, it was found that the straightforward numerical approach is only suitable as long as the heat-transfer coefficient is constant; then the initial temperature level is set to the integrated average of the fluid temperature (see the discussion in Section 2). With a variable heat-transfer coefficient we do not know the temperature at $Y \to \infty$ so that this method will not be successful.

The scheme which was eventually adopted is basically similar to the one discussed recently in connection with the transient problem of the quenching of a solid sphere [24]. It evaluates the first few cycles very rapidly so that a wrong initial temperature level can be adjusted before substantial time is wasted. The problem, given by equations (1)–(3), is Laplace-transformed with respect to time (see Appendix B) and solved in terms of the unknown surface heat-flux. The inversion via the convolution integral yields a non-linear Volterra integral equation of the form

$$T_s(t) = \sqrt{\left(\frac{\alpha}{\pi k^2}\right)} \int_0^t q_s(t-\nu)\frac{1}{\sqrt{\nu}}\,\mathrm{d}\nu. \tag{13}$$

This is inserted in the boundary condition (2). A forward-marching process is then developed by splitting the integral into suitable time intervals Δt and assuming that within such an interval the flux q_s or its mean value is independent of time. Thereby the integral can be solved and the final relation for evaluating the surface flux at time $(n\Delta t)$ becomes

$$q_{s,n} = (hT_g)_n - \frac{h_n}{k}\sqrt{\left(\frac{\alpha\Delta t}{\pi}\right)} \sum_{j=1}^{n} \times \{(q_{s,n+1-j} - q_{s,n-j})[\sqrt{j} - \sqrt{(j-1)}]\}. \tag{14}$$

Once the flux is known one determines the surface temperature at $(n\Delta t)$ from the original boundary condition (2). The important sections of two different profiles $\theta_s(\tau)$ at $\phi = \pi/6$ are plotted in Fig. 6; in one case the heat-transfer coefficient is constant whereas in the other the transfer coefficient during heating is ten times as large as during cooling. The latter case would represent approximately the situation in the brick lining of a rotary kiln.

The advantage of this scheme is that the space coordinate has been eliminated whereby the necessary calculation effort is reduced substantially. On the other hand, the nature of the convolution integral implies that the flux at each point in time be calculated from all the previous values down to the start. We effectively calculate the transient behaviour and periodicity is introduced only by changing the values of h and T_g at multiples of the cycle time. In addition, the complete series in equation (14) must be evaluated for each increase in n because at each time the previous values of q_n are multiplied by different weighting factors. Therefore, the calculation time blows up dramatically for a large absolute time, i.e. number of cycles. Still, we found that once the correct starting level of temperature has been established—this can be done by including certain heuristic steps in the computer program—the periodic steady state, within limits of accuracy, is reached after a maximum of ten cycles.

Each cycle was split into 60 and under extreme conditions into 120 time intervals; the latter was necessary for large changes in temperature over a small period of time. Then the average flux $q_{s,n}$ as defined in Appendix B is becoming increasingly inaccurate (see Fig. 6, top curve). The phase angle of heating was varied such that

$$\frac{\phi}{2\pi} = \tfrac{1}{12}; \tfrac{1}{6}; \tfrac{1}{4}; \tfrac{1}{2}; \tfrac{3}{4}$$

and within each value of ϕ the ratio of heat-transfer coefficients was chosen as

$$h_2/h_1 = 1; 2; 3; 4; 5; 10.$$

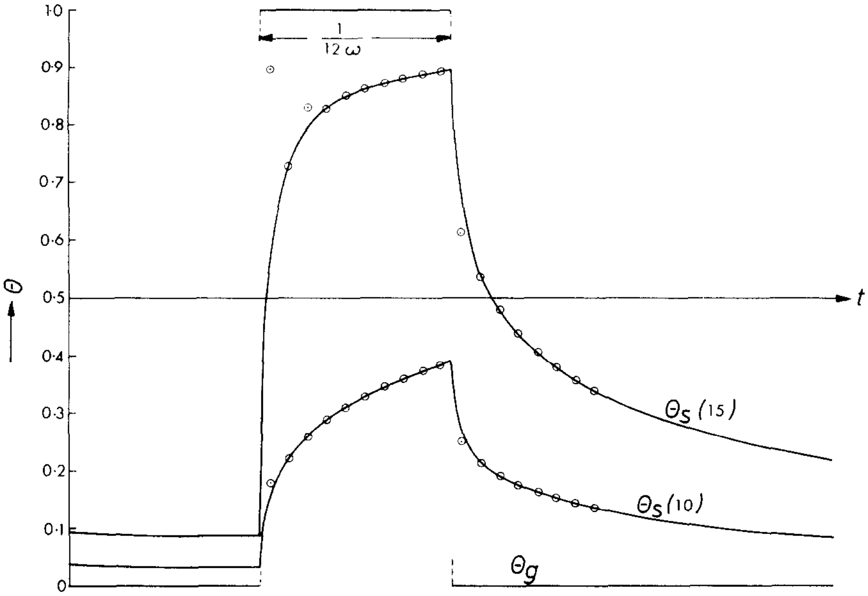

FIG. 6. Numerically evaluated temperature profiles on the solid surface; the numbers refer to those in Table 1.

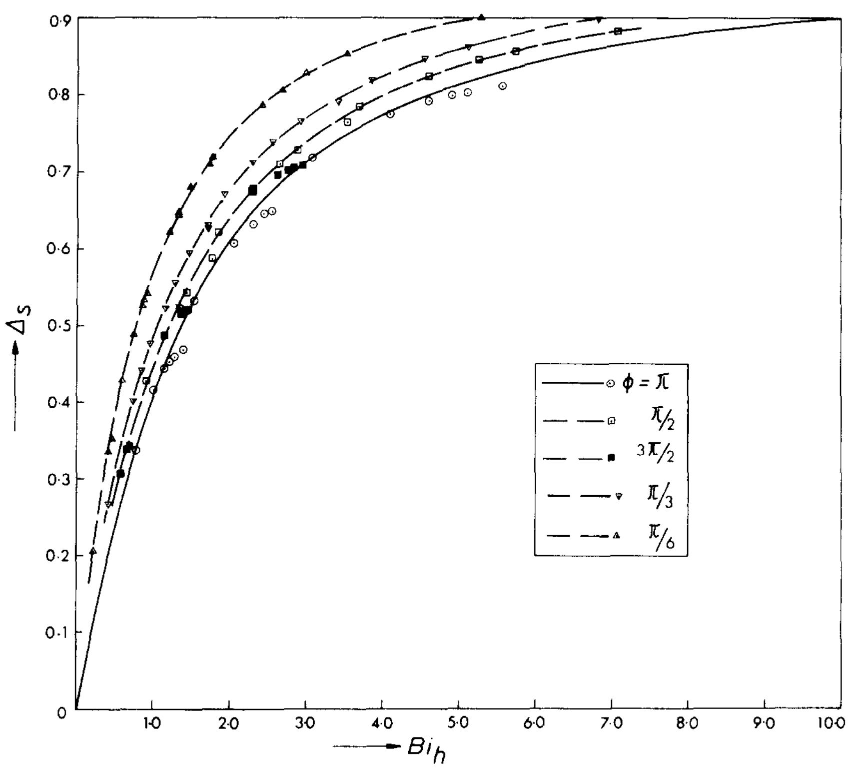

FIG. 7. Correlation of the numerical results for Δ_s on the basis of a time-weighted harmonic average of the heat-transfer coefficient.

Together with a constant value of

$$\sqrt{(\alpha/\omega)}/k = 0.0765$$

and heat-transfer coefficients

$$10 \leqslant h_1 \leqslant 40$$
$$10 \leqslant h_2 \leqslant 400$$

the covered range of Biot numbers becomes the same as in Figs. 3–5. It may be pointed out that ratios of $h_2/h_1 < 1$ are covered as well due to the symmetry of the problem, e.g.

$$\Delta_s, H(h_2/h_1 = 0.2; \phi/2\pi = 0.25) = \Delta_s, H(h_2/h_1 = 5; \phi/2\pi = 0.75).$$

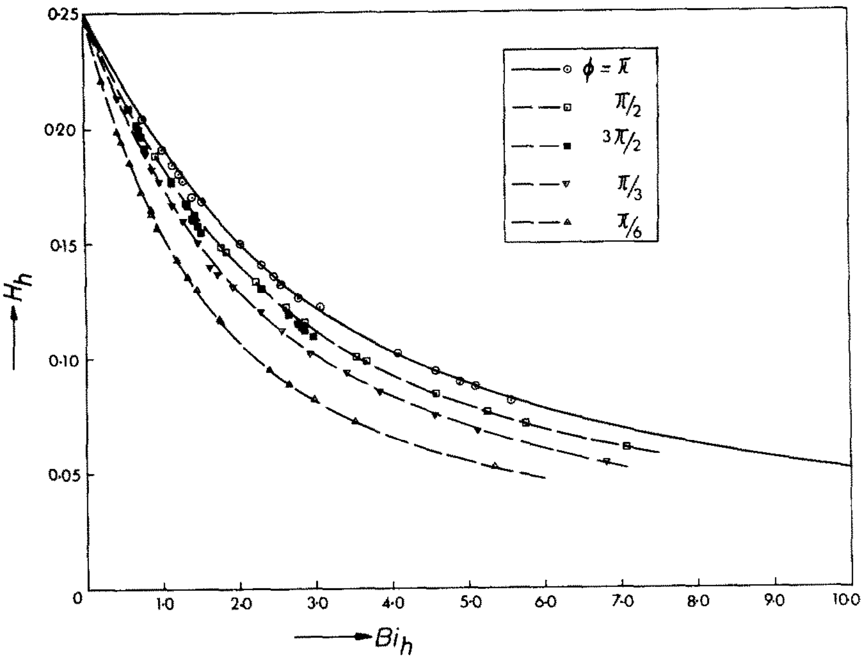

FIG. 8. Correlation of the numerical results for H on the basis of a time-weighted harmonic average of the heat-transfer coefficient.

Table 1. Extract of numerical results for variable h_1, h_2 and ϕ

No	ϕ	h_2	h_1	Θ_s	Δ_s	H.h	Bi	Bi_h	H_h	Bi_h*
10	$\pi/6$	20	20	0.210	0.3534	1.194	1.530	0.469	0.195	0.848
11		40		0.315	0.5255	1.844	-	0.863	0.164	1.561
12		60		0.375	0.6212	2.238	-	1.201	0.143	2.173
13		80		0.410	0.6809	2.489	-	1.470	0.130	2.659
14		100		0.435	0.7183	2.661	-	1.750	0.117	3.173
15		200		0.491	0.8056	3.096	-	2.673	0.089	4.836
20	$\pi/3$	20	20	0.288	0.4377	2.028	1.530	0.850	0.183	1.140
21		40		0.396	0.5956	2.875	-	1.457	0.151	1.955
22		60		0.448	0.6706	3.310	-	1.913	0.132	2.567
23		80		0.479	0.7119	3.581	-	2.267	0.121	3.041
24		100		0.498	0.7381	3.758	-	2.550	0.113	3.421
25		200		0.539	0.7897	4.154	-	3.400	0.094	4.562
30	$\pi/2$	20	20	0.349	0.4853	2.636	1.530	1.148	0.176	1.325
31		40		0.452	0.6221	3.514	-	1.836	0.146	2.120
32		60		0.500	0.6793	3.935	-	2.295	0.131	2.650
33		80		0.525	0.7091	4.177	-	2.623	0.122	3.028
34		100		0.540	0.7278	4.330	-	2.869	0.116	3.313
35		200		0.572	0.7634	4.661	-	3.531	0.101	4.077
40	π	20	20	0.500	0.5338	3.373	1.530	1.530	0.169	1.530
41		40		0.580	0.6086	3.988	-	2.040	0.150	2.040
42		60		0.610	0.6333	4.230	-	2.295	0.141	2.295
43		80		0.624	0.6452	4.360	-	2.448	0.136	2.448
44		100		0.635	0.6486	4.418	-	2.550	0.133	2.550
45		200		0.650	0.6608	4.582	-	2.782	0.126	2.782
50	$3\pi/2$	20	20	0.651	0.4853	2.636	1.530	1.148	0.176	1.325
51		40		0.700	0.5111	2.837	-	1.311	0.166	1.514
52		60		0.715	0.5171	2.902	-	1.377	0.161	1.590
53		80		0.722	0.5193	2.934	-	1.412	0.159	1.631
54		100		0.726	0.5205	2.950	-	1.434	0.157	1.656
55		200		0.733	0.5212	2.984	-	1.481	0.154	1.710

The results are all displayed in Figs. 7 and 8, to be discussed later. However, there they are already hidden in a specifically processed form so that it may be useful to present at least a representative extract of numerical results as obtained from equation (14). This has been done in Table 1 where the variation of $\bar{\theta}_s$, Δ_s and q with ϕ and ratio of heat-transfer coefficient can be followed numerically. For $h_2/h_1 = 1$ the results agree well with the analytic ones from Section 3; but it would certainly be convenient if all the other results could be correlated by some average heat-transfer coefficient for use in the analytic solution. This is attempted in the following section with the corresponding numerical results included in Table 1.

5. AVERAGE HEAT-TRANSFER COEFFICIENT

In the following we have to keep in mind our basic aim of correlating the numerical results such that effectively a closed-form solution to the general problem is obtained. However, lacking an exact solution we have to compromise between simplicity and accuracy of the final relations. With the scheme of Section 4 a specific problem can be solved to any degree of accuracy but when one comes to, e.g. optimizing such periodic processes, a simple "analytic" form of the result is required which only needs to be of a sufficient relative accuracy.

It was mentioned in Section 3 that neither the harmonic nor the exponential average of the heat-transfer coefficient would suffice to project all the curves of Figs. 3 and 5 onto those of the symmetric problem $\phi = \pi$. In the present study these averages are defined as (see [3]).

$$h_h = \frac{2}{\dfrac{\pi}{h_2\phi} + \dfrac{\pi}{h_1(2\pi-\phi)}} \tag{15}$$

and

$$h_e = 4k\sqrt{\left(\frac{\omega}{\alpha}\right)}\tanh^{-1}\left(\frac{1}{\dfrac{1}{\exp(Bi_1)-1} + \dfrac{1}{1-\exp(-Bi_2)}}\right) \tag{16}$$

with

$$Bi_1 = \frac{h_1(2\pi-\phi)}{2\pi k}\sqrt{(\alpha/\omega)};\quad Bi_2 = \frac{h_2\phi}{2\pi k}\sqrt{(\alpha/\omega)}. \tag{17}$$

For constant h equation (15) reduces to

$$h_h = h\frac{\phi(2\pi-\phi)}{\pi^2}; \tag{18}$$

i.e. a harmonic average of the relative phase times; alternatively one may obtain an exponential average from equation (16). It is easily verified that by either of the two formulae the variation of H and Δ_s with ϕ becomes far too large. Only in the limit of $Bi \to 0$, where equations (15) and (16) become identical, both averages hold exactly; this is evident from equation (11) which in combination with equation (18) gives

$$H_{Bi\to 0} = 0.25 \text{ (for all } \phi \text{ and } h_1, h_2).$$

Further, a variation in h_2/h_1 is no more interchangeable with the corresponding change in ϕ as was the case in [3]. This is expected from purely physical considerations since time ϕ and quality h of heat transfer have different effects on the temperature distribution in the solid. Therefore, we may as well first correlate the results with variable h and do a separate correlation of the parameter ϕ thereafter.

All the numerical results were processed on the basis of equations (15) and (16); it is pointed out that ϕ merely acts as a time-weighting factor on h and cannot be correlated adequately itself. Yet it is remarkable that variable ratios of heat-transfer coefficients can be correlated with high accuracy. This may be somewhat surprising since both equations (7) and (12) are not linearly dependent on h. Over the covered range of ratios and absolute values of h_1 and h_2 equation (16) holds within a maximum error of 2% whereas equation (15) leads to a maximum deviation of 5% from the analytic ($\phi = \pi$) or a smooth curve ($\phi \gtrless \pi$). For simplicity of the results we preferred to display the results from equation (15) (see Figs. 7 and 8) and may remark that trends of error, as seen for $\phi = \pi$ and $\phi = 3\pi/2$, are virtually eliminated by the use of equation (16). This is because the harmonic approximation becomes less meaningful for larger ratios and values of Bi_1 and Bi_2 (see also [3]).

The second part of the correlation refers to the phase angle of heating. By effectively applying equation (18) we have overemphasized the influence of ϕ so that now the curves with $\phi \lesseqgtr \pi$ lie on the other side of the symmetric curve $\phi = \pi$. Hence, it is sensible to try a correction factor such that

$$Bi_h^* = Bi_h\left(\frac{\pi^2}{\phi(2\pi-\phi)}\right)^n.$$

In fact, it turns out that without any significant loss in accuracy $n = 0.5$ so that

$$Bi_h^* = \frac{h_h}{k}\sqrt{\left[\frac{\alpha}{\omega}\cdot\frac{\pi^2}{\phi(2\pi-\phi)}\right]}. \tag{19}$$

The quality of the final correlations is seen in Figs. 9 and 10 where the solid curves represent the simplified equations (7) and (12), i.e.

$$\Delta_s \equiv \frac{T_{s,\max} - T_{s,\min}}{T_{g2} - T_{g1}} = \frac{4}{Bi_h^*\sqrt{\pi}}\sum_{n=1,3,5\ldots}' \times\left\{\frac{1}{(\sqrt{n})[1 + 2\sqrt{(\pi n)}/Bi_h^* + 2\pi n/(Bi_h^*)^2]}\right\} \tag{20}$$

and

$$H \equiv \frac{q}{h_h(T_{g2} - T_{g1})} = \frac{2}{\pi^{3/2}Bi_h^*}\sum_{n=1,3,5\ldots}^{\infty} \times\left\{\frac{1 + 2\sqrt{(\pi n)}/Bi_h^*}{n^{3/2}[1 + 2\sqrt{(\pi n)}/Bi_h^* + 2\pi n/(Bi_h^*)^2]}\right\}. \tag{21}$$

Thus, all the assymmetry of the problem is expressed in the parameters given by equations (15) and (19) and the evaluation of Δ_s and H becomes remarkably simple.

It is recalled that for the actual calculation of extreme temperatures we also need the arithmetic average of the extreme temperatures. For a constant heat-transfer coefficient this was given by equation (8) or Fig. 4 but for different heat-transfer coefficients during heating and cooling changes are expected. However, once an

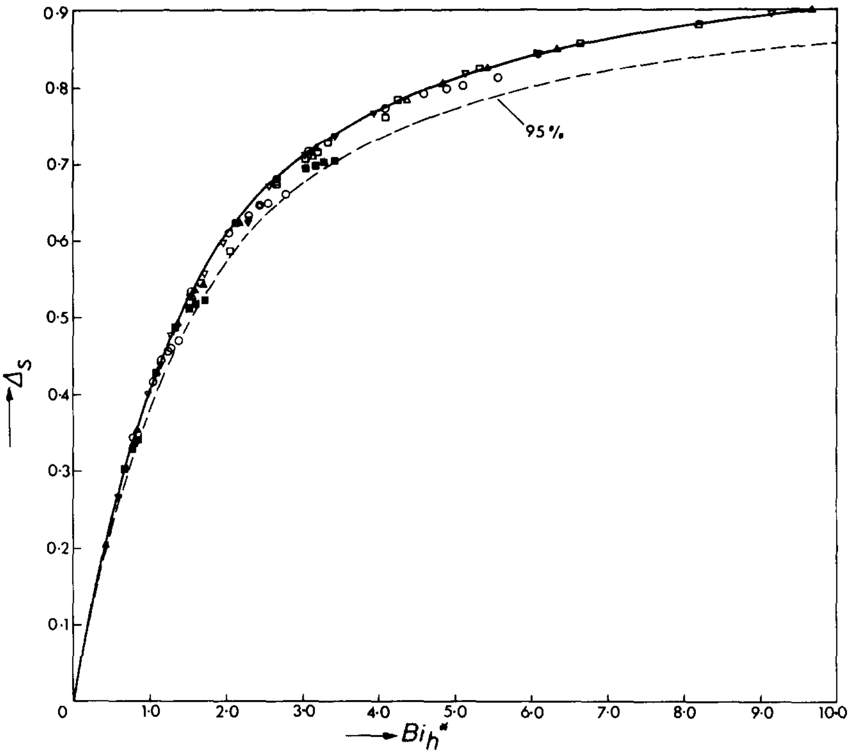

FIG. 9. Final correlation of Δ_s; the solid curve represents the analytic solution of the symmetric problem.

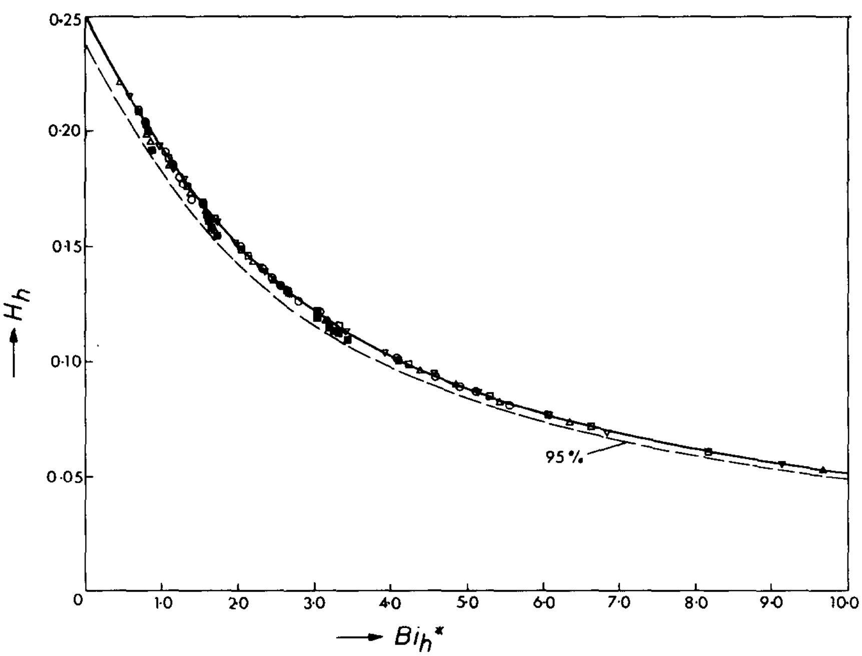

FIG. 10. Final correlation of H; the solid curve represents the analytic solution of the symmetric problem.

approximation has been established by which the assymmetric problem is reduced to a symmetric one then the same procedure should be applicable to both the difference and the sum of the extreme temperatures. Hence, we can adopt the same method as in [3]: different heat-transfer coefficients h_1 and h_2 are replaced by a corresponding change in ϕ such that Bi_1 and Bi_2 as well as their ratio are the same before and after the operation. Thereby we obtain a new, constant value for h and the Biot number is calculated from equation (19),

$$Bi^* = \frac{h_{\text{new}}}{k}\sqrt{\left(\frac{\alpha}{\omega}\right)}\sqrt{\left(\frac{\phi(2\pi-\phi)}{\pi^2}\right)}. \qquad (19a)$$

Here, ϕ is the new artificial value of the phase angle and the correction factor arises from the fact that ϕ is correlated by the square root of the harmonic average (see above). In equation (8), too, one has to employ the new value of ϕ which just compensates for the different heat-transfer coefficients. For illustration a few values from Table 1 are included in Fig. 4 and it is seen that the same degree of accuracy is achieved as in Figs. 9 and 10.

This completes a simple and yet physically-based correlation and we believe that in view of the complexity of the original problem a fairly accurate approximation has been developed. If higher accuracy was ever required either the concept of exponential average could be used or a numerical solution to the specific problem obtained along the described method.

6. CONCLUSIONS

The problem of periodic heat exchange between a fluid and a thermally thick solid body has been studied under conditions of practical interest. The step change in fluid temperature, though not exactly possible in a real process, often provides a first approximation which is safe with respect to extreme temperatures and accurate as far as the storage capacity is concerned. However, the major goal was to account for different heat-transfer coefficients during heating and cooling without having to go through a numerical procedure for each individual problem. Therefore an analytic solution for equal heat-transfer coefficients was developed and a suitable average devised from the numerical results to asymmetric problems. This aforegoing has the advantage of supplying a closed-form solution to the general problem with the ratios of heat-transfer coefficients and phase times being expressed in a single parameter. A particularly simple result is obtained by using the time-weighted harmonic average of the heat-transfer coefficients and the square root of the harmonic average of the relative phase times of heating and cooling. In comparison with numerically determined values of extreme temperatures and storage capacity the error may reach 5% in extremely asymmetric cases but would still be sufficiently small for most purposes; by employing more elaborate schemes one can, however, improve the accuracy.

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APPENDIX A

According to the method of complex temperature [12] we have to generate a new problem such that all the cos-terms of the old problem (see equation (2)) are converted into sin-terms. This is achieved by the following shift:

$$\cos\left[2\pi n\left(\tau-\frac{2n+1}{4n}\right)\right]=(-1)^n\sin(2\pi n\tau). \quad (A.1)$$

Hence the new problem is given by

$$\frac{\partial\theta^*}{\partial\tau}=\frac{\partial^2\theta^*}{\partial Y^2}, \quad (A.2)$$

$$\frac{\partial\theta_s^*}{\partial Y}=Bi\left[\theta_s^*-\frac{2}{\pi}\sum_{n=1}^{\infty}\left\{(-1)^n\frac{\sin(n\phi/2)}{n}\sin(2\pi n\tau)\right\}\right] \quad (A.3)$$

and

$$\theta^*|_{Y=\infty}=0. \quad (A.4)$$

Now the complex temperature is defined as

$$\psi=\theta+i\theta^* \quad (A.5)$$

whereby the complex problem becomes

$$\frac{\partial\psi}{\partial\tau} = \frac{\partial^2\psi}{\partial Y^2} \tag{A.6}$$

with boundary conditions

$$\frac{\partial\psi_s}{\partial Y} = Bi\left[\psi_s - \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{\sin(n\phi/2)}{n}\exp((-1)^n i2\pi n\tau)\right\}\right] \tag{A.7}$$

and

$$\psi|_{Y=\infty} = 0. \tag{A.8}$$

In order to simplify the further procedure, we may discard the term $(-1)^n$. This is feasible because we can solve any complex problem the real part of which represents the original problem. Thereby equation (A.7) becomes

$$\frac{\partial\psi_s}{\partial Y} = Bi\left[\psi_s - \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{\sin(n\phi/2)}{n}\exp(i2\pi n\tau)\right\}\right]. \tag{A.9}$$

Being only interested in the periodic steady state we set

$$\psi = \sum_{n=1}^{\infty}\left\{\Phi_n(y)\frac{\sin(n\phi/2)}{n}\exp(i2\pi n\tau)\right\}. \tag{A.10}$$

Inserting this into equation (A.6) we obtain

$$\sum_{n=1}^{\infty}\left\{\frac{\sin(n\phi/2)}{n}\exp(i2\pi n\tau)\left[i2\pi n\Phi_n - \frac{d^2\Phi_n}{dY^2}\right]\right\} = 0 \tag{A.11}$$

which for all times τ can have a non-trivial solution only if

$$\frac{d^2\Phi_n}{dY^2} - i2\pi n\Phi_n = 0. \tag{A.12}$$

The solution to this ordinary differential equation is

$$\Phi_n = B_{1,n}\exp[Y\sqrt{(i2\pi n)}] + B_{2,n}\exp[-Y\sqrt{(i2\pi n)}]. \tag{A.13}$$

Equations (A.8) and (A.10) yield

$$B_{1,n} = 0$$

so that

$$\Phi_n = B_n\exp[-Y\sqrt{(i2\pi n)}] \tag{A.14}$$

and

$$\left.\frac{d\Phi_n}{dY}\right|_{Y=0} = -B_n\sqrt{(i2\pi n)}. \tag{A.15}$$

The B_n have to be evaluated from the boundary condition (A.9) and after some complex algebra we find that

$$B_n = \frac{2}{\pi}\left(\frac{\sqrt{(\pi n)/Bi}+1-i\sqrt{(\pi n)/Bi}}{1+2\sqrt{(\pi n)/Bi}+2\pi n/Bi^2}\right). \tag{A.16}$$

Inserting equation (A.16) into (A.14) and the latter into (A.10) yields

$$\psi = \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{\sin(n\phi/2)}{n}\cdot\frac{1+\sqrt{(\pi n)/Bi}-i\sqrt{(\pi n)/Bi}}{1+2\sqrt{(\pi n)/Bi}+2\pi n/Bi^2}\right.$$
$$\left.\times\exp[-Y\sqrt{(\pi n)}]\cdot\exp\{i[2\pi n\tau - Y\sqrt{(\pi n)}]\}\right\}. \tag{A.17}$$

It may easily be verified that the real part of equation (A.17) is identical to equation (5), i.e.

$$\mathcal{R}e(\psi) = \theta(\tau, Y).$$

APPENDIX B

The Laplace transform of equation (1) is

$$\frac{\partial^2\bar{T}}{\partial y^2} - \frac{s}{\alpha}\bar{T} + \frac{s}{\alpha}T(y,0) = 0. \tag{B.1}$$

Normalizing $\bar{T}(y, s)$ on the constant initial temperature level $T(y, 0)$ yields

$$\frac{\partial^2\bar{T}}{\partial y^2} - \frac{s}{\alpha}\bar{T} = 0$$

with the solution

$$\bar{T}(y,s) = D_1(s)\exp[y\sqrt{(s/\alpha)}] + D_2(s)\exp[-y\sqrt{(s/\alpha)}]. \tag{B.2}$$

From boundary condition (3) it follows that

$$D_1(s) = 0$$

and

$$\bar{T}(y,s) = D(s)\exp[-y\sqrt{(s/\alpha)}]. \tag{B.3}$$

The heat flux in the Laplace domain is defined as

$$\bar{q}(y,s) = -k\frac{\partial\bar{T}}{\partial y},$$

hence

$$D(s) = \bar{q}(0,s)\sqrt{(\alpha/s)}/k. \tag{B.4}$$

The final solution in terms of the heat flux on the surface becomes

$$\bar{T}(y,s) = \bar{q}(0,s)\frac{1}{k}\sqrt{(\alpha/s)}\exp[-y\sqrt{(s/\alpha)}]. \tag{B.5}$$

The inversion proceeds via the convolution integral: Denoting the transform with $\mathcal{L}$ and the inverse with $\mathcal{L}^{-1}$ the following relation holds:

$$\mathcal{L}\left[\int_0^t\{q(t-\nu)\mathcal{L}^{-1}[g(s,y)]\}\,d\nu\right]$$
$$= \mathcal{L}[q(t,0)]\cdot\mathcal{L}\{\mathcal{L}^{-1}[g(s,y)]\}. \tag{B.6}$$

The RHS is identical to equation (B.5); with

$$\mathcal{L}^{-1}[g(s,y)] = \frac{1}{k}\sqrt{\left(\frac{\alpha}{\pi t}\right)}\exp[-y^2/(4\alpha t)] \tag{B.7}$$

the solid temperature becomes

$$T(y,t) = \frac{1}{k}\sqrt{\left(\frac{\alpha}{\pi}\right)}\int_0^t q_s(t-\nu)\frac{1}{\sqrt{\nu}}\exp[-y^2/(4\alpha\nu)]\,d\nu. \tag{B.8}$$

At the surface this simplifies to

$$T(0,t) \equiv T_s(t) = \frac{1}{k}\sqrt{\left(\frac{\alpha}{\pi}\right)}\int_0^t q_s(t-\nu)\frac{1}{\sqrt{\nu}}\,d\nu. \tag{B.9}$$

Thereby boundary condition (2) becomes

$$q_s(t) = hT_g - hT_s = hT_g - \frac{h}{k}\sqrt{\left(\frac{\alpha}{\pi}\right)}\int_0^t q_s(t-\nu)\frac{1}{\sqrt{\nu}}\,d\nu. \tag{B.10}$$

From this equation a forward-marching process is developed as follows:

On dropping the index s and letting time run in discrete intervals of length Δt, i.e.

$$t \to n\Delta t(0, 1, 2, \ldots j, \ldots l, \ldots n),$$

we get

$$q_n = h_n T_{gn} - \frac{h_n}{k}\sqrt{\left(\frac{\alpha}{\pi}\right)}\left\{\int_0^{\Delta t} q_n\frac{1}{\sqrt{\nu}}\,d\nu + \int_{\Delta t}^{2\Delta t} q_{n-1}\frac{1}{\sqrt{\nu}}\,d\nu + \ldots\right.$$
$$\left.\ldots + \int_{(j-1)\Delta t}^{j\Delta t} q_{n+1-j}\frac{1}{\sqrt{\nu}}\,d\nu + \ldots + \int_{(n-1)\Delta t}^{n\Delta t} q_1\frac{1}{\sqrt{\nu}}\,d\nu\right\}; \tag{B.11}$$

here the subscript n denotes the value of q, h or T_g during the nth interval. The surface flux during the jth interval is approximated by

$$\frac{q_{n-j+1}+q_{n-j}}{2}$$

which proved to be accurate enough for our purpose. As it is assumed to be independent of time equation (B.11) can be integrated so that

$$q_n = h_n T_{g,n} - \frac{h_n}{k}\sqrt{\left(\frac{\alpha\Delta t}{\pi}\right)}\sum_{j=1}^{n}$$
$$\times\{(q_{n+1-j}+q_{n-j})[\sqrt{j}-\sqrt{(j-1)}]\}. \tag{B.12}$$

In explicit form this becomes

$$q_n = \frac{h_n}{1+\frac{h_n}{k}\sqrt{\left(\frac{\alpha\Delta t}{\pi}\right)}}\left[T_{g,n}-\frac{1}{k}\sqrt{\left(\frac{\alpha\Delta t}{\pi}\right)}\right.$$

$$\left.\times\left(q_{n-1}+\sum_{j=2}^{n}\{(q_{n+1-j}+q_{n-j})[\sqrt{j}-\sqrt{(j-1)}]\}\right)\right]. \quad \text{(B.13)}$$

The starting values at the beginning of the heating period are

$q_0 = h_2 T_{g2}$ and q_1 from equation (B.12).

Some care has to be taken at the switching point ($l\Delta t$). Assuming a switching from T_{g2} to T_{g1} (and h_2 to h_1) one obtains the surface temperature at the end of the last "old" interval from

$$T_l = T_{g2} - q_l/h_2. \quad \text{(B.14)}$$

Then, at time (not interval) $l\Delta t$ a new heat flux q_{l+} is evaluated,

$$q_{l+} = h_1(T_{g1}-T_l) \quad \text{(B.15)}$$

which has to be used in the mean value of q_{l+1}. The procedure may be clarified by writing the terms around the switching point:

$$q_{l+2} = \frac{h_1}{1+\frac{h_1}{k}\sqrt{\left(\frac{\alpha\Delta t}{\pi}\right)}}$$

$$\times\left[T_{g1}-\frac{1}{k}\sqrt{\left(\frac{\alpha\Delta t}{\pi}\right)}\{q_{l+1}+(q_{l+1}+q_{l+})[(\sqrt{2})-1]\right.$$

$$+(q_l+q_{l-1})(\sqrt{3}-\sqrt{2})+\ldots$$

$$\left.\ldots+(q_1+q_0)[\sqrt{(l+2)}-\sqrt{(l+1)}]\}\right]. \quad \text{(B.16)}$$

Hence, depending on whether the considered interval is left or right of the switch-over point, we have to employ two different values for the heat flux.

SUR LE COEFFICIENT DE TRANSFERT MOYEN DANS UN ECHANGE PERIODIQUE DE CHALEUR: SOLIDE SEMI-INFINI

Résumé—Prolongement de travaux antérieurs, cette étude décrit l'échange de chaleur périodique entre un gaz et un solide semi-infini. La phase gazeuse subit un changement discontinu de température, la valeur du saut étant pris comme variable indépendante. Une condition à la limite de troisième espèce est imposée et on effectue une résolution analytique du problème dans le cas d'un coefficient de transfert de chaleur indépendant du temps.

Une situation plus proche de la réalité, dans laquelle le coefficient de transfert varie avec la température du gaz, est alors résolue par une méthode de progression numérique pas à pas. Ces résultats numériques sont corrélés sur la base d'un coefficient de transfert thermique moyen convenable employé dans la solution analytique. Par suite, les résultats peuvent aisément être adaptés aux techniques nouvelles destinées à optimiser les processus d'échange périodique.

ÜBER DEN MITTLEREN WÄRMEÜBERGANGSKOEFFIZIENTEN BEI PERIODISCHEM WÄRMEAUSTAUSCH: HALB-UNENDLICHE SPEICHERMASSE

Zusammenfassung—In Erweiterung einer früheren Arbeit wird der periodische Wärmeaustausch zwischen einem Gas und einer thermisch dicken Wand untersucht. Die Gastemperatur ändert sich sprungartig, und das Verhältnis von Heiz- zu Kühlzeit ist variabel. Unter einer Randbedingung dritter Art wird eine geschlossene Lösung entwickelt, wobei jedoch ein zeitlich konstanter Wärmeübergangskoeffizient vorausgesezt werden muß.

Der realistischere Fall, daß der Wärmeübergangskoeffizient von der Gastemperatur abhängt, d.h. zeitlich veränderlich ist, wird mit einem Schrittverfahren gelöst. Die Korrelation der numerischen Ergebnisse erfolgt über einen geeeigneten Mittelwert des Wärmeübergangskoeffizienten, der in die analytische Lösung eingesetzt werden kann. Damit erhält man eine geschlossene Form der Lösung, die sich leicht in Optimierungsverfahren für periodische Austauschprozesse weiterverarbeiten läßt.

О СРЕДНЕМ КОЭФФИЦИЕНТЕ ТЕПЛООБМЕНА ПРИ ПЕРИОДИЧЕСКОМ ИЗМЕНЕНИИ ТЕМПЕРАТУРЫ: ПОЛУОГРАНИЧЕННОЕ ТВЁРДОЕ ТЕЛО

Аннотация — В данной работе, являющейся продолжением предыдущих исследований, описывается теплообмен между газом и полуограниченным твёрдым телом при периодическом изменении температуры. Газовая фаза претерпевает ступенчатое изменение температуры, а длина шага является переменной величиной. Рассамтривается граничное условие третьего рода и получено аналитическое решение задачи для случая, когда коэффициент теплообмена не зависит от времени.

Более реальный случай, когда коэффициент теплообмена изменяется с температурой газа, решается методом прямой прогонки. Эти численные результаты обобщаются с учётом соответствующего среднего коэффициента теплообмена для получения аналитического решения. Таким образом, полученные результаты могут быть использованы в дальнейших расчётах в целях оптимизации процессов теплообмена.